Perpetual Debt Valuation
Revenue Growth Rate and Debt Ratio Are Time-Dependent

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In the previous white paper [3] we assumed that the revenue growth rate and debt ratio were constants. In this white paper we will extend that debt valuation model to incorporate a time-dependent (i.e. mean reverting) revenue growth rate and debt ratio. To that end we will work through the following hypothetical problem...

Our Hypothetical Problem

We are currently standing at time zero and are tasked with determining the market value of a company's interest-bearing perpetual debt. Our go-forward model assumptions are...

Description	Value
Annualized revenue (in dollars)	1,000,000
Annual debt market yield (%)	6.00
Annual revenue growth rate - short-term rate (%)	10.00
Annual revenue growth rate - long-term rate (%)	3.50
Ratio of debt to revenue - short-term ratio	0.35
Ratio of debt to revenue - long-term ratio	0.20
Transition half-life (in years)	3.00

Question 1: What is the book value of debt at time zero?

Question 2: What is the market value of debt at time zero assuming that the debt coupon rate is 5.25%?

Question 3: What is the market value of debt at time zero assuming that the debt coupon rate is 6.00%?

Mean Reversion [1]

We will define the variable λ to be the rate of mean reversion, which is the rate at which the short-term rate transitions to the long-term rate over time. The equation for the periodic rate at time t from the perspective of time zero is...

$$\text{Rate at time } t = \text{Long-term rate} + \left(\text{Short-term rate} - \text{Long-term rate}\right) \text{Exp}\left\{ -\lambda\, t \right\} \tag{1}$$

Note that the limit of Equation (1) above as time goes to infinity is...

$$\lim_{t \to \infty} \text{Rate at time } t = \text{Long-term rate} \;\; \text{...because...} \;\; \lim_{t \to \infty} \text{Exp}\left\{ -\lambda\, t \right\} = 0 \tag{2}$$

In the equations above we defined the variable λ to be the rate of mean reversion. To calibrate λ we will choose some future point in time (time = T) where the periodic rate is halfway between the short-term rate and the long-term rate (i.e. the transition half-life). The equation to calibrate λ is therefore...

$$\text{if... } \text{Exp}\left\{ -\lambda \times T \right\} = 0.50 \;\; \text{...then... } \lambda = -\frac{\ln(0.50)}{T} \tag{3}$$

Mean-Reverting Equations [2]

The base equation for a mean-reverting process where the variable t is time in years is...

$$f(t)=\int_m^n \mathrm{Exp}\left\{d+c\,t-a\,\mathrm{Exp}\left\{-\,b\,t\right\}\right\}\delta t\ \ \text{where...}\ \ a>0\ \ ,\ \ b>0\ \ ,\ \ c<0\ \ ,\ \ n>m \tag{4}$$

The solution to the base equation where $\Gamma(x,y)$ is the incomplete gamma function is...

$$f(t)=\mathrm{Exp}\left\{d\right\}a^{\frac{c}{b}}b^{-1}\left[\Gamma\left(-\frac{c}{b},\,a\,\mathrm{Exp}\left\{-\,b\,n\right\}\right)-\Gamma\left(-\frac{c}{b},\,a\,\mathrm{Exp}\left\{-\,b\,m\}\right)\right] \tag{5}$$

To simplify the calculations that follow, we will make the following equation definitions...

$$\begin{aligned}
E_1&=\mathrm{Exp}\left\{\frac{\Delta(N)}{\lambda}+\omega_L\,t-\frac{\Delta(N)}{\lambda}\,\mathrm{Exp}\left\{-\,\lambda\,t\right\}\right\}\\
E_2&=\mathrm{Exp}\left\{\frac{\Delta(N)}{\lambda}+(\omega_L-\lambda)\,t-\frac{\Delta(N)}{\lambda}\,\mathrm{Exp}\left\{-\,\lambda\,t\right\}\right\}\\
E_3&=\mathrm{Exp}\left\{\frac{\Delta(N)}{\lambda}+(\omega_L-2\,\lambda)\,t-\frac{\Delta(N)}{\lambda}\,\mathrm{Exp}\left\{-\,\lambda\,t\right\}\right\}
\end{aligned} \tag{6}$$

We will define the variable κ to be the risk-adjusted discount rate. The discounted version of Equation (6) above is...

$$\begin{aligned}
E_{1a}&=E_1\,\mathrm{Exp}\left\{-\,\kappa\,t\right\}=\mathrm{Exp}\left\{\frac{\Delta(N)}{\lambda}+(\omega_L-\kappa)\,t-\frac{\Delta(N)}{\lambda}\,\mathrm{Exp}\left\{-\,\lambda\,t\right\}\right\}\\
E_{2a}&=E_2\,\mathrm{Exp}\left\{-\,\kappa\,t\right\}=\mathrm{Exp}\left\{\frac{\Delta(N)}{\lambda}+(\omega_L-\lambda-\kappa)\,t-\frac{\Delta(N)}{\lambda}\,\mathrm{Exp}\left\{-\,\lambda\,t\right\}\right\}\\
E_{3a}&=E_3\,\mathrm{Exp}\left\{-\,\kappa\,t\right\}=\mathrm{Exp}\left\{\frac{\Delta(N)}{\lambda}+(\omega_L-2\,\lambda-\kappa)\,t-\frac{\Delta(N)}{\lambda}\,\mathrm{Exp}\left\{-\,\lambda\,t\right\}\right\}
\end{aligned} \tag{7}$$

Annualized Revenue [3]

We will define the variable ω_S to be the continuous-time, short-term revenue growth rate, the variable ω_L to be the continuous-time, long-term revenue growth rate, and the variable ω_t to be the continuous-time revenue growth rate at time t. Using Equation (1) above, the equation for the revenue growth rate at time t is...

$$\omega_t=\omega_L+\Delta(\omega)\,\mathrm{Exp}\left\{-\,\lambda\,t\right\}\ \ \text{...where...}\ \ \Delta(\omega)=\omega_S-\omega_L \tag{8}$$

We will define the variable Γ_t to be the cumulative revenue growth rate over the time interval $[0,t]$. Using Equation (8) above, the equation for the cumulative revenue growth rate at time t is... [1]

$$\Gamma_t=\int_0^t \omega_s\,\delta s=\frac{\Delta(\omega)}{\lambda}+\omega_L\,t-\frac{\Delta(\omega)}{\lambda}\,\mathrm{Exp}\left\{-\,\lambda\,t\right\} \tag{9}$$

We will define the variable R_t to be annualized revenue at time t. Using Equation (9) above, the equation for annualized revenue at time t as a function of annualized revenue at time zero is...

$$R_t=R_0\,\mathrm{Exp}\left\{\Gamma_t\right\}=R_0\,\mathrm{Exp}\left\{\frac{\Delta(\omega)}{\lambda}+\omega_L\,t-\frac{\Delta(\omega)}{\lambda}\,\mathrm{Exp}\left\{-\,\lambda\,t\right\}\right\} \tag{10}$$

The equation for the derivative of Equation (10) above with respect to time is...

$$\frac{\delta R_t}{\delta t}=R_0\,\mathrm{Exp}\left\{\frac{\Delta(\omega)}{\lambda}+\omega_L\,t-\frac{\Delta(\omega)}{\lambda}\,\mathrm{Exp}\left\{-\,\lambda\,t\right\}\right\}\left(\omega_L+\Delta(\omega)\,\mathrm{Exp}\left\{-\,\lambda\,t\right\}\right) \tag{11}$$

Debt Balance [3]

We will define the variable ϵ_S to be the short-term ratio of debt to revenue, the variable ϵ_L to be the long-term ratio of debt to revenue, and the variable ϵ_t to be the ratio of debt to revenue at time t. The equation for the ratio of debt to revenue at time t is...

$$\epsilon_t = \epsilon_L + \Delta(\epsilon)\,\mathrm{Exp}\left\{-\lambda\, t\right\} \text{ ...where... } \Delta(\epsilon) = \epsilon_S - \epsilon_L \tag{12}$$

The equation for the derivative of Equation (12) above with respect to time is...

$$\frac{\delta\,\epsilon_t}{\delta t} = -\lambda\,\Delta(\epsilon)\,\mathrm{Exp}\left\{-\lambda\, t\right\} \text{ ...such that... } \delta\,\epsilon_t = -\lambda\,\Delta(\epsilon)\,\mathrm{Exp}\left\{-\lambda\, t\right\}\delta t \tag{13}$$

We will define the variable D_t to be the perpetual debt balance at time t. Using Equations (10) and (12) above, the equation for debt balance at time t is...

$$D_t = \epsilon_t\,R_t = R_0\,\mathrm{Exp}\left\{\Gamma_t\right\}\left(\epsilon_L + \Delta(\epsilon)\,\mathrm{Exp}\left\{-\lambda\, t\right\}\right) \tag{14}$$

The equation for the derivative of Equation (14) above via the chain rule is...

$$\frac{\delta\,D_t}{\delta t} = \frac{\delta\,\epsilon_t}{\delta t}\,R_t + \frac{\delta R_t}{\delta t}\,\epsilon_t \text{ ...such that... } \delta D_t = \left(\frac{\delta\,\epsilon_t}{\delta t}\,R_t + \frac{\delta R_t}{\delta t}\,\epsilon_t\right)\delta t \tag{15}$$

Using Equations (6), (10), (11), (12) and (13) above, we can rewrite Equation (15) above as...

$$\delta\,D_t = R_0\left[\epsilon_L\,\omega_L\,E_1 + \left(\epsilon_L\,\Delta(\omega) + (\omega_L - \lambda)\,\Delta(\epsilon)\right)E_2 + \Delta(\epsilon)\,\Delta(\omega)\,E_3\right]\delta t \tag{16}$$

We will define the variable $P_{a,b}$ to be the cumulative change in debt over the time interval $[a, b]$. Using Equation (16) above, the equation for the cumulative change in debt is...

$$P_{a,b} = \int_a^b \delta D_t = R_0\left[\epsilon_L\,\omega_L \int_0^T E_1\,\delta t + \left(\epsilon_L\,\Delta(\omega) + (\omega_L - \lambda)\,\Delta(\epsilon)\right)\int_0^T E_2\,\delta t + \Delta(\epsilon)\,\Delta(\omega)\int_0^T E_3\,\delta t\right] \tag{17}$$

We will define the variable $\bar{P}_{0,T}$ to be the present value of the cumulative change in debt over the time interval $[0, T]$. Using Equation (17) above, the equation for the present value of the cumulative change in debt is...

$$\bar{P}_{0,T} = \int_0^T \delta D_t\,\mathrm{Exp}\left\{-\kappa\, t\right\} = R_0\left[\epsilon_L\,\omega_L \int_0^T E_{1a}\,\delta t + \left(\epsilon_L\,\Delta(\omega) + (\omega_L - \lambda)\,\Delta(\epsilon)\right)\int_0^T E_{2a}\,\delta t + \Delta(\epsilon)\,\Delta(\omega)\int_0^T E_{3a}\,\delta t\right] \tag{18}$$

Debt Service [3]

We will define the variable C_t to be the annualized, pre-tax debt service at time t and the variable α to be the debt service coupon rate. Using Equation (14) above, the equation for the annualized debt service is...

$$C_t = \alpha\,D_t = \alpha\,R_0\,\mathrm{Exp}\left\{\Gamma_t\right\}\left(\epsilon_L + \Delta(\epsilon)\,\mathrm{Exp}\left\{-\lambda\, t\right\}\right) \tag{19}$$

We will define the variable $C_{a,b}$ to be cumulative, pre-tax debt service over the time interval $[a, b]$. Using Equation (19) above, the equation for cumulative debt service is...

$$C_{a,b} = \int_a^b C_t\,\delta t = \alpha\,R_0\left[\epsilon_L \int_a^b E_1\,\delta t + \Delta(\epsilon)\int_a^b E_2\,\delta t\right] \tag{20}$$

We will define the variable $\bar{C}_{0,T}$ to be the present value of pre-tax debt service payments made over the time interval $[0, T]$ and the variable κ to be the risk-adjusted discount rate (i.e. market debt yield). Using Equation (20) above, the equation for the present value of debt service is...

$$\bar{C}_{0,T} = \int_0^T C_t\,\mathrm{Exp}\left\{-\kappa\, t\right\}\delta t = \alpha\,R_0\left[\epsilon_L \int_0^T E_{1a}\,\delta t + \Delta(\epsilon)\int_0^T E_{2a}\,\delta t\right] \tag{21}$$

Answers To Our Hypothetical Problem

Using Equation (3) above and our go-forward model assumptions, the value of the model parameter λ is...

$$\lambda = -\frac{\ln(0.50)}{3.00} = 0.2310 \tag{22}$$

Using our go-forward model assumptions, the continuous-time short-term and long-term revenue growth rates are...

$$\omega_S = \ln\Bigg(1 + 0.1000\Bigg) = 0.0953 \text{ and } \omega_L = \ln\Bigg(1 + 0.0350\Bigg) = 0.0344 \text{ and } \Delta(\omega) = 0.0953 - 0.0344 = 0.0609 \tag{23}$$

Using our go-forward model assumptions, the values of our debt ratio model parameters are..

$$\epsilon_S = 0.3500 \text{ and } \epsilon_L = 0.2000 \text{ and } \Delta(\epsilon) = 0.3500 - 0.2000 = 0.1500 \tag{24}$$

The discount rate in this case is equal to the market yield on debt. Using our go-forward model assumptions above, the value of the model parameters κ and α (debt service rate) is...

$$\kappa = 0.0600 \text{ and } \alpha = 0.0525 \tag{25}$$

Using Equations (22), (23) and (25) above, the values of the exponential integrals used in the equations above are... [4]

$$\int_0^T E_{1a}\,\delta t = \text{EXPINT}(1,\ 0.0600,\ 0.2310,\ 0.0953,\ 0.0344,\ 0{,}0) = 49.5983$$

$$\int_0^T E_{2a}\,\delta t = \text{EXPINT}(2,\ 0.0600,\ 0.2310,\ 0.0953,\ 0.0344,\ 0{,}0) = 4.4270$$

$$\int_0^T E_{3a}\,\delta t = \text{EXPINT}(3,\ 0.0600,\ 0.2310,\ 0.0953,\ 0.0344,\ 0{,}0) = 2.2360 \tag{26}$$

Question 1: What is the book value of debt at time zero?

Using Equations (14) and (24) above, the answer to the question is...

$$D_0 = 0.35 \times 1,000,000 = 350,000 \tag{27}$$

Question 2: What is the market value of debt at time zero assuming that the debt coupon rate is 5.25%?

The answer to the question is...

$$V_0 = \text{PV Debt Service} - \text{PV Debt Change} = 555,645 - 285,023 = 270,622 \tag{28}$$

Using Equation (21) above and model parameter Equations (22) to (26) above, the present value of debt service in Equation (31) above is...

$$\text{PV Debt Service} = 0.0525 \times 1,000,000 \times \Bigg[0.2000 \times 49.5983 + 0.1500 \times 4.4270\Bigg] = 555,645 \tag{29}$$

Using Equation (18) above and model parameter Equations (22) to (26) above, the present value of the cumulative change in debt in Equation (31) above is...

$$\text{PV Debt Change} = 1,000,000 \times \Bigg[0.2000 \times 0.0344 \times 49.5983 + (0.2000 \times 0.0609 + (0.0344 - 0.2310) \times 0.1500)) \times 4.4270 + 0.1500 \times 0.0609 \times 2.2360\Bigg] = 285,023 \tag{30}$$

Question 3: What is the market value of debt at time zero assuming that the debt coupon rate is 6.00%?

Using the answer to Question 2 as our guide, the answer to the question is...

$$V_0 = \text{PV Debt Service} - \text{PV Debt Change} = 635,023 - 285,023 = 350,000 \tag{31}$$

Note that when the debt service coupon rate equals the discount rate (i.e. market yield) the market value of the debt equals the book value debt.

References

[1] Gary Schurman, *The Stochastic, Mean-Reverting Short Rate,* May, 2020

[2] Gary Schurman, *Incomplete Gamma Function - Base Equation For A Mean-Reverting Process,* November, 2017

[3] Gary Schurman, *Perpetual Debt Valuation - Revenue Growth Rate and Debt Ratio Are Constants,* March, 2025

[4] Gary Schurman, *Solving The Exponential Integral - Excel VBA Toolbox,* December, 2024